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Sato's Bäcklund transformations, additional symmetries and ASvM formula for the discrete KP hierarchy

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Abstract

Two kinds of symmetries, Sato's Bäcklund transformations and additional symmetries, for the discrete KP (dKP) hierarchy are introduced, and the ASvM formula which demonstrates the equivalence of these two kinds of symmetries is obtained. In this process the Fay identity and the difference Fay identity of the dKP hierarchy are introduced and the ASvM formula in the form of tau function is calculated.

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1. Introduction

Symmetries and their algebra structures [1] are essential parts of the classical integrable system. They are closely related to other properties such as infinite conserved quantities, Hamiltonian structures, Bäcklund transformations, group-invariant solutions and their classification, etc. Before 1980, it was thought that the symmetry of a given soliton equation does not depend explicitly on the space variable x and the temporal variable t in a $(1 + 1)$-dimensional space. Later, however, the symmetry depending on x and t was given by several authors in the community of the soliton theory [2–5]. It is now called additional symmetry [6] by the definition of pseudo-differential operators. A remarkable fact of additional symmetry flows of the KP hierarchy [7, 8] is that they do not commute with each other although they are commutable with KP flows. In particular, the string equation, Virasoro and W constraints in the p-reduced KP hierarchy (or the p-Gelfand–Dickey hierarchy), which are used to prove Witten conjecture with $p = 2$ by Kontsevich [9], are studied extensively through the additional symmetry flows $\partial_{t^*_{m,l}} = \partial^*_{m,l}$ by the introduction of new variables $t^*_{m,l}$ [10–14, 26]. On the

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other hand, the infinitesimal Bäcklund transformation, which is an effective way to solve the integrable hierarchy and soliton equations, also defines symmetry. The Sato's Bäcklund transformation on the *tau* function of the KP hierarchy is given by the vertex operator $X(\lambda, \mu)$ acting on *tau* with two parameters [7, 15]. These two symmetry flows seem to be independent; however, they are actually equivalent, which was demonstrated through the ASvM formula in [16, 17]. In [26], the ASvM formula, as a useful tool, helps to develop a simple method to prove D–V–V and F–K–N's conjecture which was mentioned in [27, 28]. In addition, the ASvM formula of the BKP hierarchy was given by Johan [18] and Tu [14] using different methods.

The discrete KP (dKP) hierarchy [19, 20] defined by the difference operator Δ is an interesting object for the current research of the integrable system. The Hamiltonian structures, the existence of the *tau* function and the gauge transformation operators for the dKP hierarchy are given in [19–21]. More recently, the extended discrete KP and the algebraic structures of the (non-)isospectral flows of the dKP hierarchy were investigated in [22, 23]. Note that there exists another form of the dKP hierarchy [24, 25] defined by the shift operator Γ, and some interesting results including the vertex operator, the Bäcklund transformation for this operator, have already been obtained. An advantage of using the operator $\triangle$ instead of the operator Γ is that we can use Dickey's convenient methods. Another advantage is that the definition of the tau function of the dKP hierarchy in the form of $\triangle$ has a strong connection with the tau function of the KP hierarchy. It is useful for our study. This study concentrates on the symmetries of the dKP hierarchy. Through infinitesimal operators, we introduce two kinds of symmetries, Sato's Bäcklund transformations and additional symmetries, for the dKP hierarchy. And we prove that they are indeed symmetries. In addition, we establish the ASvM formula which demonstrates the equivalence of these two kinds of symmetries. Based on that we obtain the ASvM formula in the form of tau function.

The organization of the paper is as follows. In section 2, we give a brief description of the discrete KP hierarchy and prove some useful identities of the discrete pseudo-difference operators. In section 3, we introduce Sato's Bäcklund transformation for the dKP hierarchy through vertex operators and prove that they indeed define symmetries of the dKP hierarchy. In section 4, we introduce the additional symmetries for dKP and establish the ASvM formula. In section 5, we find the action of additional symmetry flows on the *tau* functions based on the ASvM formula developed in section 4. Section 6 is devoted to conclusions and discussion.

2. The discrete KP and difference operators

To be self-contained, we give a brief introduction to the dKP hierarchy based on a detailed research in [20].

Let F be an associative ring of functions which include a discrete variable $n \in \mathbb{Z}$ and infinite time variables $t_i \in \mathbb{R}$:

$$F = \{f(n) = f(n, t_1, t_2, \ldots, t_j, \ldots); n \in \mathbb{Z}, t_i \in \mathbb{R}\}.$$

We denote the shift operator and the difference operator by Γ and $\triangle$, respectively. Their actions on F are as follows:

$$\Gamma f(n) = f(n+1)$$

and

$$\triangle f(n) = f(n+1) - f(n) = (\Gamma - I) f(n),$$

where I is the identity operator. If we consider a function $f(n)$ as an operator whose action on $g(n) \in F$ is $f(n)g(n)$, we can infer the following identity about multiplication of function operators and difference operators. That is, for any $j \in \mathbb{Z}$

$$\triangle^j \circ f = \sum_{i=0}^{\infty} \binom{j}{i} (\triangle^i f)(n+j-i)\triangle^{j-i}, \qquad \binom{j}{i} = \frac{j(j-1)\cdots(j-i+1)}{i!}. \tag{2.1}$$

Here, the multiplication is denoted by '$\circ$'. If the function operators are located on the the left-hand side, we omit '$\circ$'. So with (2.1) we can obtain an associative ring $F(\triangle)$ of formal pseudo difference operators, which includes two operations '+' and '$\circ$':

$$F(\triangle) = \left\{ R(n) = \sum_{j=-\infty}^{d} f_j(n)\triangle^j,\ f_j(n) \in F, n \in \mathbb{Z} \right\}.$$

The ring F includes two subrings which are $F_+(\triangle) = \left\{R_+ = \sum_{j=0}^{d} f_j(n)\triangle^j\right\}$, the ring of deference operators and $F_-(\triangle) = \left\{R_- = \sum_{j=-\infty}^{-1} f_j(n)\triangle^j\right\}$, the ring of Volterra operators.

We can also define the adjoint operator to $\triangle$ which is denoted by $\triangle^*$. Its action on F is

$$\triangle^* f(n) = (\Gamma^{-1} - I) f(n) = f(n-1) - f(n),$$

where $\Gamma^{-1} f(n) = f(n-1)$. The corresponding identity about multiplication '$\circ$' is

$$\triangle^{*j} \circ f = \sum_{i=0}^{\infty} \binom{j}{i} (\triangle^{*i} f)(n+i-j)\triangle^{*j-i}.$$

Then we also obtain the adjoint ring $F(\triangle^*)$ to $F(\triangle)$. The formal adjoint to $R \in F(\triangle)$ is $R^* \in F(\triangle^*)$ which is defined by $R^* = \sum_{j=-\infty}^{d} \triangle^{*j} \circ f_j(n)$. Here, the $*$ operation satisfies $(F \circ G)^* = G^* \circ F^*$ for two operators and $f(n)^* = f(n)$ for a function. If we consider $R(n)$ as a series in $\triangle$, we can define the operation of taking residue on $F(\triangle)$, that is $\mathrm{res}_{\triangle} R = f_{-1}(n)$. Then between the residue on $R(n)$ and the residue on a common series in z, there is a useful identity [20]: for $X, Y \in F(\triangle)$,

$$\mathrm{res}_z \left(X(n)(1+z)^n \exp\left(\sum_{i=1}^{\infty} t_i z^i\right)\right) \left(Y^*(n-1)(1+z)^{-n} \exp\left(-\sum_{i=1}^{\infty} t_i z^i\right)\right) = \mathrm{res}_{\triangle} XY. \tag{2.2}$$

Let L be a general first-order pseudo difference operator:

$$L = L(n) = \triangle + \sum_{j=0}^{\infty} f_j(n)\triangle^{-j}. \tag{2.3}$$

The discrete KP hierarchy [20] can be expressed as

$$\frac{\partial L}{\partial t_i} = [(L^i)_+, L]. \tag{2.4}$$

Comparing the powers of $\triangle$ on both sides, we can obtain a family of evolution equations in functions $f_i(n)$. Define the dressing operator

$$W(n;t) = 1 + \sum_{j=1}^{\infty} w_j(n;t)\triangle^{-j},$$

which satisfies

$$L = W \circ \triangle \circ W^{-1}. \tag{2.5}$$

Then the Sato equation in the operator W

$$\frac{\partial W}{\partial t_i} = (L^i)_- \circ W \tag{2.6}$$

is equivalent to the dKP equation (2.4). The dKP hierarchy can also be expressed as the following equations equivalently:

$$L^k w = z^k w \qquad \text{and} \qquad \partial_{t_m} w = L_+^m w, \tag{2.7}$$

and

$$(L^*(n-1))^k w^* = z^k w^* \qquad \text{and} \qquad \partial_{t_m} w^* = -(L^*(n-1))_+^m w^*. \tag{2.8}$$

Note that here (2.8) is different from the continuous case of the KP hierarchy. Using the dressing operator W, we can give a kind of form solutions for the above equations which are called Baker or wavefunctions $w(n;t,z)$ and adjoint Baker or wavefunctions $w^*(n;t,z)$:

$$\begin{aligned} w(n;t,z) &= W(n;t)(1+z)^n \exp\left(\sum_{i=1}^{\infty} t_i z^i\right) \\ &= \left(1 + \frac{w_1(n;t)}{z} + \frac{w_2(n;t)}{z^2} + \cdots\right)(1+z)^n \exp\left(\sum_{i=1}^{\infty} t_i z^i\right) \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} w^*(n;t,z) &= (W^{-1}(n-1;t))^*(1+z)^{-n} \exp\left(\sum_{i=1}^{\infty} -t_i z^i\right) \\ &= \left(1 + \frac{w_1^*(n;t)}{z} + \frac{w_2^*(n;t)}{z^2} + \cdots\right)(1+z)^{-n} \exp\left(\sum_{i=1}^{\infty} -t_i z^i\right). \end{aligned} \tag{2.10}$$

Considering these and using Sato equation (2.6), we can verify that (2.7) and (2.8) hold. Wavefunctions and adjoint wavefunctions satisfy the following bilinear identity and the corresponding inverse.

Proposition 2.1 ([20]). *For any $j \geqslant 0$ and for any multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ where $\partial^\alpha = \partial_{t_1}^{\alpha_1}\partial_{t_2}^{\alpha_2}\cdots\partial_{t_k}^{\alpha_k}$ and $\alpha_k \geqslant 0$,*

$$\mathrm{res}_z(\triangle^j \partial^\alpha w(n;t,z))w^*(n;t,z) = 0. \tag{2.11}$$

Correspondingly, given two formal series

$$\begin{aligned} w(n;t,z) &= (1+z)^n\left(1 + \frac{w_1(n;t)}{z} + \frac{w_2(n;t)}{z^2} + \cdots\right)\exp\left(\sum_{i=1}^{\infty} t_i z^i\right) \\ &= z^n(1+O(z^{-1}))\exp\left(\sum_{i=1}^{\infty} t_i z^i\right) \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} w^*(n;t,z) &= (1+z)^{-n}\left(1 + \frac{w_1^*(n;t)}{z} + \frac{w_2^*(n;t)}{z^2} + \cdots\right)\exp\left(-\sum_{i=1}^{\infty} t_i z^i\right) \\ &= z^{-n}(1+O(z^{-1}))\exp\left(-\sum_{i=1}^{\infty} t_i z^i\right) \end{aligned} \tag{2.13}$$

satisfying the above bilinear identity (2.11), $w(n;t,z)$ and $w^*(n;t,z)$ are necessarily the wavefunctions and the adjoint wavefunctions of the dKP hierarchy, respectively.

A dKP hierarchy is equivalent to a single function, the tau function $\tau_\triangle = \tau(n;t)$ [20], which means that all functions $w_i(n)$ in the dressing operator W can be generated by a single function, the tau function. That is

$$w(n;t,z) = \frac{\tau(n;t-[z^{-1}])}{\tau(n;t)}(1+z)^n \exp\left(\sum_{i=1}^{\infty} t_i z^i\right) \tag{2.14}$$

and

$$w^*(n;t,z) = \frac{\tau(n;t+[z^{-1}])}{\tau(n;t)}(1+z)^{-n} \exp\left(\sum_{i=1}^{\infty} -t_i z^i\right), \tag{2.15}$$

where $[z] = \left(z, \frac{z^2}{2}, \frac{z^3}{3}, \ldots\right)$. And further all functions $f_i(n)$ in the operator L can be generated by the tau function too [21]. Here introduce $G(z)$ and $G^*(z)$ operators, whose actions on functions are $G_t(z)f(t) = f(t-[z^{-1}])$ and $G^*_{t'}(z)f(t') = f(t'+[z^{-1}])$, respectively.

There is a useful tool given by Dickey.

Proposition 2.2 ([8]). *If*

$$f(z) = \sum_{-\infty}^{\infty} a_i(\zeta) z^{-i},$$

then

$$\operatorname{res}_z\left(\left(\zeta^{-1}\left(1-\frac{z}{\zeta}\right)^{-1} + z^{-1}\left(1-\frac{\zeta}{z}\right)^{-1}\right) f(z)\right) = f(\zeta), \tag{2.16}$$

where $\left(1-\frac{z}{\zeta}\right)^{-1}$ *is understood as a series in* ζ^{-1} *and* $\left(1-\frac{\zeta}{z}\right)^{-1}$ *is a series in* z^{-1}.

Now we prove a useful property for the difference operators.

Lemma 2.3. *Let* $R \in F(\triangle)$*; then*

$$R(n)_- = \sum_{i=1}^{\infty} \triangle^{-i} \circ \operatorname{res}_\triangle(\triangle^{i-1} \circ R(n+1)). \tag{2.17}$$

Proof. Note that $R(n)$ has another expression:

$$R(n) = \sum_{i=-\infty}^{d} f_i(n)\triangle^i = \sum_{i=-\infty}^{d} \triangle^i \circ g_i(n), \tag{2.18}$$

and

$$R(n)_- = \sum_{i=-\infty}^{-1} f_i(n)\triangle^i = \sum_{i=-\infty}^{-1} \triangle^i \circ g_i(n). \tag{2.19}$$

Taking residue on both sides of (2.18),

$$\operatorname{res}_\triangle \sum_{i=-\infty}^{-1} f_i(n)\triangle^i = f_{-1}(n) = \operatorname{res}_\triangle \sum_{i=-\infty}^{-1} \triangle^i \circ g_i(n) = g_{-1}(n-1). \tag{2.20}$$

Here we use the second expression of $R(n)$ to prove identity (2.17). On the right-hand side of (2.17), $\operatorname{res}_\triangle \triangle^{i-1} \circ R(n+1)_+ = 0$; then we only need to consider the contribution of $R_-(n+1)$. On the other hand, due to the linear property of the operations $\operatorname{res}_\triangle$ and $\triangle$, we just need to consider a general entry in $R_-(n+1)$. Without loss of any generality take the entry in the form of $\triangle^{-k} \circ g_{-k}(n+1)$ for any $k \in \mathbb{Z}_+$. So we have

$$\sum_{i=1}^{\infty} \triangle^{-i} \circ \mathrm{res}_{\triangle}(\triangle^{i-1} \circ \triangle^{-k} \circ g_{-k}(n+1)) = \triangle^{-k} \circ \mathrm{res}_{\triangle}\triangle^{-1} \circ g_{-k}(n+1) = \triangle^{-k} g_{-k}(n). \tag{2.21}$$

Combining with the linear property, the lemma is proved. □

The special case of the above lemma is that for $n = 1$

$$f(n) \circ \triangle^{-1} = \sum_{i \geqslant 0} \triangle^{-i-1} \circ \triangle^{i}(f(n+1)). \tag{2.22}$$

This result has already been proved in [21].

3. Vertex operators and Sato's Bäcklund transformations

In this section, we consider a kind of symmetries for the dKP hierarchy which are called Sato's Bäcklund transformations. Since dKP can be generated by tau functions, we just need to consider the symmetries on $\tau_\triangle$. By the definition of the Lie algebra we need to find the infinitesimal operators $X(n; \lambda, \mu)$ of these symmetries which act on tau functions. And through solving the equations $\frac{\partial \tau_\triangle}{\partial t^*_{\lambda,\mu}} = X(n; \lambda, \mu)\tau_\triangle$ where $t_{\lambda,\mu}$ are the variables about these symmetries, we can obtain for each value of $t_{\lambda,\mu}$ a new tau function. That is, we obtain the symmetries marked by $t_{\lambda,\mu}$.

Define the infinitesimal operators $X(n; \lambda, \mu)$ on the space of tau functions $\tau_\triangle$ as follows:

$$X(n; \lambda, \mu) = \left(\frac{1+\mu}{1+\lambda}\right)^n : \exp \sum_{-\infty}^{\infty} \left(\frac{P_i}{\mathrm{i}\lambda^i} - \frac{P_i}{\mathrm{i}\mu^i}\right) :, \tag{3.1}$$

where

$$P_i = \begin{cases} \partial_i, & i > 0 \\ |i| t_{|i|}, & i \leqslant 0 \end{cases}$$

and the symbol of normal ordering '::' means that all P_i with negative i should be placed on the left of positive ones. Note that $X(n; \lambda, \mu)$ are vertex operators. By direct computation, the action of $X(n; t, z)$ on $\tau_\triangle$ functions is

$$X(n; \lambda, \mu)\tau(n; t) = \left(\frac{1+\mu}{1+\lambda}\right)^n \exp\left(\sum_{1}^{\infty} t_i(\mu^i - \lambda^i)\right) G^*(\lambda)G(\mu)\tau(n; t). \tag{3.2}$$

Now, we prove that $X(n; \lambda, \mu)$ are indeed the infinitesimal operators of the symmetries of Sato's Bäcklund transformations, which is the following theorem.

Theorem 3.1. *If ϵ is infinitesimal and $\tau(n; t)$ is a tau function of the dKP hierarchy, $\tilde{\tau}(n; t) = \tau(n; t) + \epsilon X(n; \lambda, \mu)\tau(n; t)$ is also a tau function of the dKP hierarchy.*

Proof. Let

$$\tilde{w}(n; t, z) = \frac{G(z)(\tau(n; t) + \epsilon X(n; \lambda, \mu)\tau(n; t))}{\tau(n; t) + \epsilon X(n; t, z)\tau(n; t)}(1+z)^n \exp \sum_{i=1}^{\infty} t_i z^i \tag{3.3}$$

and

$$\tilde{w}^*(n;t',z) = \frac{G^*(z)(\tau(n;t') + \epsilon X(n;\lambda,\mu)\tau(n;t'))}{\tau(n;t') + \epsilon X(n;t',z)\tau(n;t')}(1+z)^{-n} \exp \sum_{i=1}^{\infty} -t_i' z^i \tag{3.4}$$

be functions generated by $\tilde{\tau}(n;t)$. If we want to prove that $\tilde{\tau}(n;t)$ is a new *tau* function generated by $X(n;\lambda,\mu)$, we just need to verify that $\tilde{w}(n;t,z)$ and $\tilde{w}^*(n;t,z)$ are the wavefunction and the adjoint wavefunction of the dKP hierarchy, respectively. Based on the second part of proposition 2.1, we only need to verify for $j \geqslant 0$:

$$\mathrm{res}_z(\triangle^j \tilde{w}(n;t,z)\tilde{w}^*(n;t',z)) = 0.$$

Further it is equivalent to verify for $m \geqslant n$

$$\mathrm{res}_z(\tilde{w}(m;t,z)\tilde{w}^*(n;t',z)) = 0. \tag{3.5}$$

Substitute (3.3) into the above equation. Note that $\tau(n;t)$ is a tau function of dKP and it satisfies

$$\mathrm{res}_z(G(z)\tau(m;t))(G^*(z)\tau(n;t))(1+z)^{m-n} \exp \sum_{1=1}^{\infty}(t_i - t_i')z^i = 0. \tag{3.6}$$

So by the definition of $O(\epsilon)$, we just need to verify that the following identity holds for $m \geqslant n$:

$$\mathrm{res}_z(G_t(z)X(m;\lambda,\mu)\tau(m,t))(G_{t'}^*(z)\tau(n;t'))(1+z)^{m-n} \exp \sum_{1=1}^{\infty}(t_i - t_i')z^i$$
$$+\,\mathrm{res}_z(G_t(z)\tau(m;t))(G_{t'}^*(z)X'(n;\lambda,\mu)\tau(n;t')(1+z)^{m-n} \exp \sum_{1=1}^{\infty}(t_i - t_i')z^i = 0, \tag{3.7}$$

where $X'(n;\lambda,\mu)$ denotes $X(n;\lambda,\mu)$ with t replaced by t'. Substitute (3.2) into (3.7), and the first item of the above equation is

$$\mathrm{res}_z\, G_t(z)G_t^*(\lambda)G_t(\mu)\tau(m;t)\cdot G_{t'}^*(z)\tau(n;t')$$
$$\cdot (1+z)^{m-n}\left(\frac{1+\mu}{1+\lambda}\right)^m \exp \sum_{1=1}^{\infty}(t_i z^i - t_i' z^i - t_i\lambda^i + t_i\mu^i)\cdot \frac{1-\mu/z}{1-\lambda/z}. \tag{3.8}$$

On the other hand, by (2.11), we have

$$\mathrm{res}_z\, G_t(z)\tau(m;t)\cdot G_{t'}^*(z)\tau(n;t')(1+z)^{m-n} \exp \sum_{1=1}^{\infty}(t_i - t_i')z^i = 0. \tag{3.9}$$

Let $G_t^*(\lambda)G_t(\mu)$ act on both sides of the above identity; then we obtain

$$\mathrm{res}_z\, G_t^*(\lambda)G_t(\mu)G_t(z)\tau(m;t)\cdot G_{t'}^*(z)\tau(n;t')(1+z)^{m-n} \exp \sum_{1=1}^{\infty}(t_i - t_i')z^i \cdot \frac{1-z/\mu}{1-z/\lambda} = 0. \tag{3.10}$$

Multiply the above by

$$\frac{\mu}{\lambda} \exp \sum_{1=1}^{\infty}(t_i\mu^i - t_i\lambda^i)\cdot\left(\frac{1+\mu}{1+\lambda}\right)^m,$$

and subtract it from (3.8); then (3.8) becomes

$$\mathrm{res}_z\, G_t(z)G_t^*(\lambda)G_t(\mu)\tau(m;t)\cdot G_{t'}^*(z)\tau(n;t')(1+z)^{m-n}\left(\frac{1+\mu}{1+\lambda}\right)^m$$
$$\cdot\exp\sum_{1=1}^{\infty}(t_iz^i-t_i'z^i-t_i\lambda^i+t_i\mu^i)\cdot(z-\mu)\left(\frac{1}{z(1-\lambda/z)}+\frac{1}{\lambda(1-z/\lambda)}\right). \qquad (3.11)$$

Using proposition 2.2, the above expression which is the first item of (3.7) equals

$$\mathrm{res}_z\, G_t(\mu)\tau(m;t)\cdot G_{t'}^*(\lambda)\tau(n;t')\frac{(1+\mu)^m}{(1+\lambda)^n}\exp\sum_{1=1}^{\infty}(t_i\mu^i-t_i'\lambda^i)\cdot(\lambda-\mu). \qquad (3.12)$$

Similarly, the second item of (3.7) equals

$$\mathrm{res}_z\, G_t(z)\tau(m;t)\cdot G_{t'}^*(z)G_{t'}^*(\lambda)G_{t'}(\mu)\tau(n;t')$$
$$\cdot(1+z)^{m-n}\left(\frac{1+\mu}{1+\lambda}\right)^n\exp\sum_{1=1}^{\infty}(t_iz^i-t_i'z^i-t_i'\lambda^i+t_i'\mu^i)\cdot\frac{1-\lambda/z}{1-\mu/z}. \qquad (3.13)$$

To get (3.12)+(3.13) = 0, we further simplify the above expression. So let $G_{t'}^*(\lambda)G_{t'}(\mu)$ act on (3.9); then we have

$$\mathrm{res}_z\, G_t(z)\tau(m;t)\cdot G_{t'}^*(z)G_{t'}^*(\lambda)G_{t'}(\mu)\tau(n;t')(1+z)^{m-n}\exp\sum_{1=1}^{\infty}(t_i-t_i')z^i\cdot\frac{1-z/\lambda}{1-z/\mu}=0. \qquad (3.14)$$

Multiply the above by

$$\frac{\lambda}{\mu}\exp\sum_{1=1}^{\infty}t_i'\mu^i-t_i'\lambda^i\cdot\left(\frac{1+\mu}{1+\lambda}\right)^n$$

and subtract it from (3.13); then (3.13) becomes

$$\mathrm{res}_z\, G_t(z)\tau(m;t)\cdot G_{t'}^*(\lambda)G_{t'}(\mu)G_{t'}^*(z)\tau(n;t')(1+z)^{m-n}\left(\frac{1+\mu}{1+\lambda}\right)^n$$
$$\cdot\exp\sum_{1=1}^{\infty}(t_iz^i-t_i'z^i-t_i'\lambda^i+t_i'\mu^i)\cdot(z-\lambda)\left(\frac{1}{z(1-\mu/z)}+\frac{1}{\mu(1-z/\mu)}\right). \qquad (3.15)$$

So based on proposition 2.2, the above expression which is the second item of (3.7) becomes

$$\mathrm{res}_z\, G_t(\mu)\tau(m;t)\cdot G_{t'}^*(\lambda)\tau(n;t')\frac{(1+\mu)^m}{(1+\lambda)^n}\exp\sum_{1=1}^{\infty}(t_i\mu^i-t_i'\lambda^i)\cdot(\mu-\lambda). \qquad (3.16)$$

So (3.16) cancels (3.12) and the identity of (3.7) holds. Then $\tilde\tau(n;t)$ is a new *tau* function of the dKP hierarchy, which means that $X(n;\lambda,\mu)$ are indeed infinitesimal operators on the space of $\tau(n;t)$ functions of dKP. □

Now we introduce the Fay identity and the difference Fay identity for the dKP hierarchy.

Lemma 3.2. *The tau function of the dKP hierarchy $\tau(n;t)$ satisfies the following Fay identity:*

$$\sum_{(s_1,s_2,s_3)}(s_0-s_1)(s_2-s_3)\tau(n;t+[s_0]+[s_1])\tau(n;t+[s_2]+[s_3])=0, \qquad (3.17)$$

where (s_1,s_2,s_3) means the cyclic permutation of s_1, s_2 and s_3.

Proof. Note that for a fixed n, $\tau(n;t)$ is also a tau function of KP. So it also satisfies the Fay identity of KP's, which infers (3.17). □

Note that there is a connection between $\tau(t)$ functions of the KP hierarchy and $\tau(n;t)$ functions of the dKP hierarchy, that is [20]

$$\tau_\triangle = \tau(n;t) = \tau\left(t_1 + n, t_2 - \frac{n}{2}, t_3 + \frac{n}{3}, \ldots\right). \tag{3.18}$$

So we can obtain the following difference Fay identity.

Lemma 3.3. *The tau functions $\tau(n;t)$ of the dKP hierarchy satisfy the following difference Fay identity:*

$$\begin{aligned}&\left(s_3^{-1}+1\right)\triangle\left(\frac{\tau(n;t+[s_1]-[s_3]}{\tau(n;t)}\right)\\ &\quad=\left(s_3^{-1}-s_1^{-1}\right)\frac{\tau\left(n;t-\left[s_3^{-1}\right]\right)\cdot\tau(n+1;t+[s_1])-\tau(n;t+[s_1]-[s_3])\tau(n+1;t)}{\tau(n;t)\tau(n+1;t)}.\end{aligned} \tag{3.19}$$

Proof. Consider the Fay identity (3.17). Let $s_0 = 0$; divide it by $s_1s_2s_3$ and shift the variable $t \to t - [s_2] - [s_3]$. Then we have

$$\begin{aligned}&\left(s_3^{-1}-s_2^{-1}\right)\tau(n;t+[s_1]-[s_2]-[s_3])\tau(n;t)\\ &\quad+\left(s_1^{-1}-s_3^{-1}\right)\tau(n;t-[s_3])\tau(n;t+[s_1]-[s_2])\\ &\quad+\left(s_2^{-1}-s_1^{-1}\right)\tau(n;t-[s_2])\tau(n;t+[s_1]-[s_3])\\ &=0.\end{aligned} \tag{3.20}$$

On the other hand, based on (3.18) we have the following connection:

$$\tau(n+1,t) = \tau(n;t-[-1]). \tag{3.21}$$

Then let $s_2 = -1$ in (3.20); use the above connection and we obtain the difference Fay identity (3.19). □

4. Additional symmetries and the ASvM formula

Now, we consider another family of symmetries for the dKP hierarchy which are called the additional symmetries. Similar to the case of the KP hierarchy [6, 11, 12], first define

$$\Gamma_\triangle = \sum_{i=1}^{\infty}\left(\mathrm{i}t_i\triangle^{i-1} + (-1)^{i-1}n\triangle^{i-1}\right), \tag{4.1}$$

which has the following property.

Lemma 4.1.

$$\left[\partial_{t_k} - \triangle^k, \Gamma_\triangle\right] = 0. \tag{4.2}$$

Proof.

$$\begin{aligned}\left[\partial_{t_k}-\triangle^k,\Gamma_\triangle\right]&=\left[\partial_k,\sum_{i=1}^{\infty}\mathrm{i}t_i\triangle^{i-1}\right]-\left[\triangle^k,\sum_{i=1}^{\infty}(-1)^{i-1}n\triangle^{i-1}\right]\\ &=k\triangle^{k-1}-\sum_{i=1}^{\infty}(-1)^{i-1}((n+k)\triangle^k+k\triangle^{k-1})\triangle^{i-1}+\sum_{i=1}^{\infty}(-1)^{i-1}n\triangle^{k+i-1}\end{aligned}$$

$$
\begin{aligned}
&= k\triangle^{k-1} - \sum_{i=1}^{\infty}(-1)^{i-1}k\triangle^{k+i-1} - \sum_{i=1}^{\infty}(-1)^{i-1}k\triangle^{k+i-2}\\
&= k\triangle^{k-1} - k\triangle^{k-1}\\
&= 0.
\end{aligned}
\tag{4.3}
$$

This property plays an important role in the definition of additional symmetries. In addressing (4.2), we also define another operator

$$M_\triangle = W \circ \Gamma_\triangle \circ W^{-1}. \tag{4.4}$$

These two operators are crucial blocks to construct the additional symmetries for the dKP hierarchy. Now, we demonstrate some properties about them. With respect to dKP flows (2.4), the evolution equation of $M_\triangle$ is

$$\partial_{t_k} M_\triangle = \left[L_+^k, M_\triangle\right]. \tag{4.5}$$

More generally,

$$\partial_{t_k}\left(M_\triangle^m L^l\right) = \left[L_+^k, M_\triangle^m L^l\right]. \tag{4.6}$$

The action of $M_\triangle$ on wavefunctions is calculated by

$$
\begin{aligned}
M_\triangle w &= W\Gamma_\triangle\left((1+z)^n \exp\sum_{i=1}^{\infty}(t_i z^i)\right)\\
&= W\left(\sum_{i=1}^{\infty}\left(t_i i z^i (1+z)^n \exp\sum_{j=1}^{\infty} t_j z^j + (-1)^{i-1} n z^{i-1}(1+z)^n \exp\sum_{j=1}^{\infty} t_j z^j\right)\right)\\
&= W\left((1+z)^n\left(\partial_z\left(\exp\sum_{i=1}^{\infty} t_j z^j\right)\right) + (\partial_z((1+z)^n))\exp\sum_{i=1}^{\infty} t_j z^j\right)\\
&\quad\left(\text{with}\quad (1+z)^{-1} = \sum_{i=0}^{\infty}(-1)^i z^i\right)\\
&= W\left(\partial_z\left((1+z)^n \exp\sum_{i=1}^{\infty} t_j z^j\right)\right)\\
&= \partial_z w.
\end{aligned}
\tag{4.7}
$$

So together with $Lw = zw$, we have more general formulas

$$L^l M_\triangle^m w = \partial_z^m z^l w, \qquad M_\triangle^m L^l w = z^l \partial_z^m w.$$

Moreover, a straightforward calculation can infer the following commutation relations:

$$[\triangle, \Gamma_\triangle] = 1 \qquad \text{and} \qquad [L, M_\triangle] = 1. \tag{4.8}$$

With the above calculations on the operators $\Gamma_\triangle$ and $M_\triangle$, we can now define the additional symmetries for the dKP hierarchy. For every pair of m, l, we define additional flows of the dKP hierarchy as follows:

$$\partial_{ml}^* W = -(M_\triangle^m L^l)_- \circ W, \tag{4.9}$$

where ∂_{ml}^* symbolizes a derivative with respect to an additional variable t_{ml}^*. Together with the definition of L and M, the action of additional flows in (4.9) implies that the action of additional flows on the operator L is

$$\partial_{ml}^* L = -\left[\left(M_\triangle^m L^l\right)_-, L\right] \tag{4.10}$$

and that on the operator M is

$$\partial^*_{ml} M_\triangle = -\big[\big(M^m_\triangle L^l\big)_-, M_\triangle\big]. \tag{4.11}$$

Combining both of them, we obtain a general form, that is

$$\partial^*_{ml} M^n_\triangle L^k = -\big[\big(M^m_\triangle L^l\big)_-, M^n_\triangle L^k\big], \tag{4.12}$$

for any $k \in \mathbb{Z}$ and $n \in \mathbb{Z}$.

Now we calculate commutation relations between additional flows $\partial^*_{m,l}$ and original flows ∂_{t_k} of the dKP hierarchy. □

Lemma 4.2. *The flows of $\partial^*_{m,l}$ commute with all flows of ∂_{t_k} of the dKP hierarchy.*

Proof. According to the action of ∂_{t_k} and $\partial^*_{m,l}$ on the Sato operator W, we have

$$\begin{aligned}\big[\partial^*_{ml}, \partial^*_{t_k}\big](W) &= -\partial^*_{ml}\big(L^k_- \circ W\big) + \partial_{t_k}\big(\big(M^m_\triangle L^l\big)_- \circ W\big)\\ &= \big[\big(M^m_\triangle L^l\big)_-, L^k\big]_- \circ W + L^k_- \circ \big(M^m_\triangle L^l\big)_- \circ W - \big(M^m_\triangle L^l\big)_- \circ L^k_- \circ W\\ &\quad + \big(M^m_\triangle \circ \big[L^k_+, L^l\big]\big)_- \circ W + \big(\big[L^k_+, M^m_\triangle\big] \circ L^l\big)_- \circ W\\ &= \big[\big(M^m_\triangle L^l\big)_-, L^k\big]_- \circ W + \big[L^k_+, M^m_\triangle L^l\big]_- \circ W + \big[L^k_-, \big(M^m_\triangle L^l\big)_-\big] \circ W\\ &= 0. \end{aligned} \tag{4.13}$$

Therefore, additional flows ∂^*_{ml} commute with all dKP flows ∂_{t_k}. □

Based on this lemma, if L satisfies the dKP hierarchy (2.4) and ϵ is infinitesimal, then $L + \epsilon\partial^*_{ml} L$ also satisfies the dKP hierarchy by the definition of $O(\epsilon)$:

$$\begin{aligned}\frac{\partial(L + \epsilon\partial^*_{ml} L)}{\partial t_i} &= \big[L^i_+, L\big] + \epsilon\partial^*_{ml}\big[L^i_+, L\big]\\ &= \big[L^i_+ + \epsilon(\partial^*_{ml} L^i)_+, L + \epsilon\partial^*_{ml} L\big]\\ &= \big[(L + \epsilon\partial^*_{ml} L)^i_+, L + \epsilon\partial^*_{ml} L\big]. \end{aligned} \tag{4.14}$$

This means that the additional flows which are defined by the operators ∂^*_{ml} indeed define symmetries of the dKP hierarchy. So it is called additional symmetry (flows) without any confusion. The operators ∂^*_{ml} are infinitesimal operators of additional symmetries, and the symmetries are remarked by the variables t^*_{ml}. Note that, similar to the KP case, additional symmetry flows $\partial^*_{m,l}$ do not commute with each other. We will discuss about it in the next section.

Now there are two kinds of symmetries, additional symmetries and Sato Bäcklund transformations. They are not independent but equivalent. Through the connection between ∂^*_{ml} and $X(n; \lambda, \mu)$ which are the infinitesimal operators of these two symmetries, the ASvM formula demonstrates the equivalence of two symmetries. To obtain ASvM, we first combine the infinitesimal operators ∂^*_{ml}. In the space of wavefunctions, we can define the combined operator Y which is also an infinitesimal operator of additional symmetries as follows:

$$Y(n; \lambda, \mu) = \sum_{m=0}^{\infty} \frac{(\mu - \lambda)^m}{m!} \sum_{l=-\infty}^{\infty} \lambda^{-l-m-1}\big(M^m_\triangle L^{m+l}\big)_-. \tag{4.15}$$

It has another expression.

Lemma 4.3.

$$Y(n; \lambda, \mu) = w(n; t, \mu) \circ \triangle^{-1} \circ w^*(n + 1; t, \lambda). \tag{4.16}$$

Proof. By lemma 2.3 and identity (2.2), we have

$$\begin{aligned}
\left(M_\triangle^m L^{m+l}\right)_- &= \sum_{i=1}^{\infty} \triangle^{-i} \circ \operatorname{res}_\triangle\left(\triangle^{i-1} \circ M_\triangle^m(n+1) \circ W(n+1) \circ \triangle^{m+l} \circ W^{-1}(n+1)\right) \\
&= \sum_{i=1}^{\infty} \triangle^{-i} \circ \operatorname{res}_z\Bigg(\left(\left(\triangle^{i-1} \circ M_\triangle^m(n+1) \circ W(n+1) \circ \triangle^{m+l}\right)\right. \\
&\quad \times \left.\left((1+z)^{n+1} \exp \sum_{j=1}^{\infty} t_j z^j\right)\right) \cdot \left((W^*)^{-1}(n)\left((1+z)^{-n-1} \exp \sum_{j=1} -t_j z^j\right)\right)\Bigg) \\
&= \sum_{i=1}^{\infty} \triangle^{-i} \circ \operatorname{res}_z\left(z^{m+l} \cdot \triangle^{i-1}\left(M_\triangle^m(n+1) w(n+1,;t,z)\right) \cdot w^*(n+1;t,z)\right) \\
&= \operatorname{res}_z\left(z^{m+l} \sum_{i=1}^{\infty} \triangle^{-i} \circ \triangle^{i-1}\left(\partial_z^m w(n+1;t,z)\right) \circ w^*(n+1;t,z)\right) \\
&= \operatorname{res}_z\left(z^{m+l}\left(\partial_z^m w(n;t,z) \circ \triangle^{-1} \circ w^*(n+1;t,z)\right)\right).
\end{aligned} \tag{4.17}$$

In the penultimate equality, (4.7) is used. Substitute (4.17) into the definition of Y and use proposition 2.2; then

$$\begin{aligned}
Y(n;\lambda,\mu) &= \operatorname{res}_z\left(\left(\sum_{m=0}^{\infty} \sum_{l=-\infty}^{\infty} \frac{z^{m+l}}{\lambda^{m+l+1}} \cdot \frac{(\mu-\lambda)^m}{m!}\left(\partial_z^m w(n;t,z)\right)\right) \circ \triangle^{-1} \circ w^*(n+1;t,z)\right) \\
&= \operatorname{res}_z\Bigg(\left(\left(z^{-1}\left(1-\frac{\lambda}{z}\right)^{-1} + \lambda^{-1}\left(1-\frac{z}{\lambda}\right)^{-1}\right) \cdot \exp((\mu-\lambda)\partial_z) \cdot w(n;t,z)\right) \\
&\quad \circ \triangle^{-1} \circ w^*(n+1;t,z)\Bigg) \\
&= (\exp((\mu-\lambda)\partial_\lambda) \cdot w(n;t,\lambda)) \circ \triangle^{-1} \circ w^*(n+1;t,z) \\
&= w(n;t,\mu) \circ \triangle^{-1} \circ w^*(n+1;t,\lambda).
\end{aligned} \tag{4.18}$$

□

Now the ASvM formula for the dKP hierarchy, which sets up the connection between the Sato's Bäcklund transformations and additional symmetries, is given by the following theorem.

Theorem 4.4. *The infinitesimal operator $X(n;\lambda,\mu)$ has the following connection with the infinitesimal operator $Y(n;\lambda,\mu)$ in the space of wavefunctions:*

$$X(n;\lambda,\mu) w(n;t,z) = (\lambda-\mu) Y(n;\lambda,\mu) w(n;t,z), \tag{4.19}$$

where the action of $X(n;\lambda,\mu)$ on wavefunctions should be understood as generated by its action (3.3) on the $\tau(n;t)$ function.

Proof. First, we calculate the action of $X(n;\lambda,\mu)$ on the wavefunction. We begin with the analog in the KP hierarchy [8, 17]:

$$\begin{aligned}
X(\lambda,\mu) w(t,z) &= \exp\left(\sum_{i=1}^{\infty} t_i z^i\right) X(\lambda,\mu) \frac{\tau(t-[z^{-1}])}{\tau(t)} \\
&= \left(\frac{\lambda-\mu}{z-\lambda}\right) \exp\left(\sum_{i=1}^{\infty} t_i (z^i + \mu^i - \lambda^i)\right) \frac{\tau(t-[\mu^{-1}])\tau(t-[z^{-1}]+[\lambda^{-1}])}{\tau^2(t)}.
\end{aligned} \tag{4.20}$$

Through a shift of variables

$$(t_1, t_2, \ldots) \to \left(t_1 + n, t_2 - \frac{n}{2}, t_3 + \frac{n}{3}, \ldots\right),$$

we obtain

$$\begin{aligned} X(n;\lambda,\mu)w(n;t,z) &= (1+z)^n \exp\left(\sum_{i=1}^{\infty} t_i z^i\right) X(n;\lambda,\mu)\frac{\tau(n;t-[z^{-1}])}{\tau(n;t)} \\ &= (1+z)^n \left(\frac{1+\mu}{1+\lambda}\right)^n \left(\frac{\lambda-\mu}{z-\lambda}\right) \exp\left(\sum_{i=1}^{\infty} t_i(z^i+\mu^i-\lambda^i)\right) \\ &\quad \times \frac{\tau(n;t-[\mu^{-1}])\tau(n;t-[z^{-1]}+[\lambda^{-1}])}{\tau^2(n;t)}, \\ &= (\lambda-\mu)w(n;t,\mu)\left(\frac{1+z}{1+\lambda}\right)^n \exp\left(\sum_{i=1}^{\infty} t_i(z^i-\lambda^i)\right)(z-\lambda)^{-1} \\ &\quad \times \frac{\tau(n;t-[z^{-1}]-[\lambda^{-1}])}{\tau(n;t)}. \end{aligned} \tag{4.21}$$

Secondly, substituting Y given by (4.16) into (4.19), it becomes

$$X(n;\lambda,\mu)w(n;t,z) = (\lambda-\mu)w(n;t,\mu)\cdot\triangle^{-1}\cdot(w^*(n+1;t,\lambda)w(n;t,z)). \tag{4.22}$$

Furthermore, substituting (4.21) into the above, it becomes

$$\begin{aligned} &(\lambda-\mu)w(n;t,\mu)\left(\frac{1+z}{1+\lambda}\right)^n \exp\left(\sum_{i=1}^{\infty} t_i(z^i-\lambda^i)\right)(z-\lambda)^{-1}\frac{\tau(n;t-[z^{-1}]-[\lambda^{-1}])}{\tau(n;t)} \\ &\quad = (\lambda-\mu)w(n;t,\mu)\cdot\triangle^{-1}\cdot(w^*(n+1;t,\lambda)w(n;t,z)). \end{aligned} \tag{4.23}$$

Divide both sides by $(\lambda-\mu)w(n;t,\mu)$ and let the operator $\triangle$ act on it. Then what we need to prove is the following identity:

$$\begin{aligned} &\triangle\left(\left(\frac{1+z}{1+\lambda}\right)^n \exp\left(\sum_{i=1}^{\infty} t_i(z^i-\lambda^i)\right)(z-\lambda)^{-1}\frac{\tau(n;t-[z^{-1}]-[\lambda^{-1}])}{\tau(n;t)}\right) \\ &\quad = \frac{\tau(n+1;t+[\lambda^{-1}])\tau(n;t-[z^{-1}])}{\tau(n+1;t)\tau(n;t)} \exp\left(\sum_{i=1}^{\infty} t_i(z^i-\lambda^i)\right)\cdot\frac{(1+z)^n}{(1+\lambda)^{n+1}}, \end{aligned} \tag{4.24}$$

which also equals

$$\begin{aligned} &(z-\lambda)\frac{\tau(n+1;t+[\lambda^{-1}])\tau(n;t-[z^{-1}])}{\tau(n+1;t)\tau(n;t)} \\ &\quad = (1+z)\frac{\tau(n+1;t-[z^{-1}]-[\lambda^{-1}])}{\tau(n+1;t)} - (1+\lambda)\frac{\tau(n;t-[z^{-1}]-[\lambda^{-1}])}{\tau(n;t)}. \end{aligned} \tag{4.25}$$

The above identity holds because it is nothing but the difference Fay identity with $s_1 = \lambda^{-1}$ and $s_3 = z^{-1}$. So (4.19) holds. □

5. Algebraic structure of additional symmetries for dKP

In this section we further concentrate on the algebraic structures of these symmetries and their action on *tau* functions and wavefunctions. The ASvM formula, in addition to constructing the connection between Sato's Bäcklund transformations and additional symmetries, also constructs the connection between their algebraic structure.

First, we demonstrate the algebraic structure of $\partial^*_{m,l}$ acting on the operator L or W. By the definition of the additional symmetries of the dKP hierarchy in the previous sections, we obtain that it is the same as the corresponding algebraic structure of the KP hierarchy.

Theorem 5.1. *For* $m, k, h, l \in \mathbb{Z}$

$$[\partial^*_{k,m+k}, \partial^*_{l,h+l}] = \sum_{i=1}^{\infty}\left(-\binom{m+k}{i}\binom{l}{i} + \binom{h+l}{i}\binom{k}{i}\right) i!\partial^*_{k+l-i,m+h+k+l-i}, \tag{5.1}$$

and when $k = 1, l = 1$,

$$[\partial^*_{1,m+1}, \partial^*_{1,h+1}] = (h-m)\partial^*_{1,m+l+1}. \tag{5.2}$$

In order to obtain $\partial^*_{m,l}\tau(n; t, z)$ by means of ASvM, we need to consider the Taylor expansion of $X(n; \lambda, \mu)$ in μ at the point of λ:

$$X(n; \lambda, \mu) = \sum_{m=0}^{\infty} \frac{(\mu-\lambda)^m}{m!} \sum_{l=-\infty}^{\infty} \lambda^{-l-m} \tilde{W}_l^{(m)}, \tag{5.3}$$

where

$$\sum_{l=-\infty}^{\infty} \lambda^{-l-m} \tilde{W}_l^{(m)} = \partial_\mu^m\big|_{\mu=\lambda} X(n; \lambda, \mu).$$

The first few items of $\tilde{W}_l^{(m)}$ are

$$\begin{aligned}
\tilde{W}_l^{(0)} &= \delta_{l,0},\\
\tilde{W}_l^{(1)} &= \tilde{P}_l,\\
\tilde{W}_l^{(2)} &= \sum_{i+j=l} : \tilde{P}_i\tilde{P}_j : -(l+1)\tilde{P}_l,\\
\tilde{W}_l^{(3)} &= \sum_{i+j+k=l} : \tilde{P}_i\tilde{P}_j\tilde{P}_k : -\frac{3}{2}(l+2)\sum_{i+j=l} : \tilde{P}_i\tilde{P}_j : +(l+1)(l+2)\tilde{P}_l,
\end{aligned} \tag{5.4}$$

where

$$\tilde{P}_i = \begin{cases} \partial_i, & i > 0\\ 0, & i = 0\\ -it_{-i} + (-1)^{-i-1}n, & i < 0. \end{cases}$$

Based on the ASvM formula in the form of wavefunctions (4.19), the Taylor expansion of Y (4.15) and X given by (5.3), we can introduce another expression of the ASvM formula which is in the form of *tau* function. It can be regarded as a discrete analog of the original ASvM formula [16] of the KP hierarchy.

Theorem 5.2. *For* $m \geqslant 0$ *and for all* l, *there is*

$$\partial^*_{m,l+m}\tau(n; t) = \frac{\tilde{W}_l^{(m+1)} \cdot \tau(n; t)}{m+1}. \tag{5.5}$$

Proof. Note

$$Y(\partial^*) = \sum_{m=0}^{\infty} \frac{(\mu-\lambda)^m}{m!} \sum_{l=-\infty}^{\infty} \lambda^{-l-m-1}\partial^*_{m,m+l},$$

Substituting wavefunctions $w(n; t, z)$ given by (2.14) into the ASvM formula (4.19), we obtain

$$X(n; \lambda, \mu)\frac{G(z)\tau(n; t)}{\tau(n; t)} = (\lambda - \mu)Y(n; \lambda, \mu)\frac{G(z)\tau(n; t)}{\tau(n; t)},$$

$$\frac{\tau(n; t) \cdot G(z)(X(n; t, z)\tau(n; t)) - (G(z)\tau(n; t)) \cdot X(n; t, z)\tau(n; t)}{\tau^2(n; t)}$$
$$= (\lambda - \mu)Y(\partial^*)\frac{G(z)\tau(n; t)}{\tau(n; t)},$$

$$(G(z) - 1)\frac{X(n; \lambda, \mu)\tau(n; t)}{\tau(n; t)} \cdot \frac{G(z)\tau(n; t)}{\tau(n; t)}$$
$$= (G(z) - 1)\frac{(\lambda - \mu)Y(\partial^*)\tau(n; t)}{\tau(n; t)} \cdot \frac{G(z)\tau(n; t)}{\tau(n; t)}.$$

So we have

$$X(n; \lambda, \mu)\tau(n; t) = (\lambda - \mu)Y(\partial^*)\tau(n; t) + c\tau(n; t),$$

which can be expressed as

$$\left(1 + \sum_{m=1}^{\infty}\frac{(\mu - \lambda)^m}{m!}\sum_{l=-\infty}^{\infty}\lambda^{-l-m}\tilde{W}_l^{(m)}\right) \cdot \tau(n; t)$$
$$= \sum_{m=0}^{\infty}\frac{(\mu - \lambda)^{m+1}}{m!}\sum_{l=-\infty}^{\infty}\lambda^{-l-m-1}\partial^*_{m,m+l} \cdot \tau(n; t) + c\tau(n; t)$$

according to Taylor expansions (4.15) and (5.3). Then, taking $c = 1$, and comparing the powers of $\mu - \lambda$ and λ on both sides, (5.5) is verified. □

Furthermore, with $\partial^*_{m,l}\tau(n; t)$ (5.5) we can obtain another expression of the action of additional symmetries on wavefunctions.

Theorem 5.3. *For $m \geqslant 0$, and $\forall\, l$,*

$$\partial^*_{m,m+l}w(n; t, z) = (G(z) - 1)\frac{\left(\tilde{W}_l^{(m+1)}/m + 1\right) \cdot \tau(n; t)}{\tau(n; t)} \cdot w(n; t, z). \tag{5.6}$$

Proof. Using $w(n; t, z)$ in (2.14) and $\partial^*_{m,l}\tau(n; t)$ in (5.5), by a straightforward calculation, we obtain

$$\partial^*_{m,m+l}w(n; t, z) = \partial^*_{m,m+l}\left(\frac{G(z)\tau(n; t)}{\tau(n; t)}(1 + z)^n \exp\sum_{i=1}^{\infty}(t_i z^i)\right)$$
$$= \frac{\tau(n; t) \cdot G(z)(\partial^*_{m,m+l}\tau(n; t)) - G(z)\tau(n; t) \cdot \partial^*_{m,m+l}\tau(n; t)}{\tau^2(n; t)}(1 + z)^n \exp\sum_{i=1}^{\infty}(t_i z^i)$$
$$= w(n; t, z)\left((G(z) - 1)\frac{\partial^*_{m,m+l}\tau(n; t)}{\tau(n; t, z)}\right)$$
$$= w(n; t, z)\left((G(z) - 1)\frac{\left(\tilde{W}_l^{(m+1)}/m + 1\right) \cdot \tau(n; t)}{\tau(n; t, z)}\right). \tag{5.7}$$

□

As a simple application of (5.6), we can obtain constraints on tau functions. For example, from

$$\partial^*_{m,l+m}w(n; t, z) = 0, \tag{5.8}$$

we have

$$(G(z)-1)\frac{\tilde{W}_l^{(m+1)}\cdot\tau(n;t)}{(m+1)\tau(n;t)}=0,$$

or equivalently,

$$\frac{\tilde{W}_l^{(m+1)}\tau(n;t)}{m+1}=c\tau(n;t). \tag{5.9}$$

Here, we would like to discuss several special cases of the above formula. First, let $m=0$; then $\partial_{0,l}^{*}w(n;t,z)=0$ implies

$$\tilde{W}_l^{(1)}\tau(n;t)=\tilde{P}_l\tau(n;t)=c\tau(n;t).$$

Here, $\tilde{P}_l$ is given by (5.4). Secondly, let $m=1$; then $\partial_{1,l+1}^{*}w(n;t,z)=0$ implies

$$\tfrac{1}{2}\tilde{W}_l^{(2)}\tau(n;t)=c\tau(n;t).$$

With the form of $\tilde{W}_l^{(2)}$ in (5.4), that is

$$\left(\frac{1}{2}\sum_{i+j=l}:\tilde{P}_i\tilde{P}_j:-(l+1)\tilde{P}_l\right)\cdot\tau(n;t)=\mathcal{L}_l\tau(n;t)=c\tau(n;t), \tag{5.10}$$

combining the results of $m=0$ and $m=1$, for any $l\in\mathbb{Z}$, we have

$$\frac{1}{2}\sum_{i+j=l}:\tilde{P}_i\tilde{P}_j:\tau(n;t)=\mathcal{L}_l\tau(n;t)=c\tau(n;t). \tag{5.11}$$

Because of

$$[\mathcal{L}_l,\mathcal{L}_m]=(l-m)\mathcal{L}_{l+m}+\delta_{l+m,0}(l^3-l)\cdot c, \tag{5.12}$$

these are just the Virasoro constraints on $\tau(n;t)$.

6. Conclusions and discussion

In this study we defined the vertex operators for dKP and introduced Sato's Bäcklund transformations for the dKP hierarchy through pseudo-difference operators. Additional symmetries were also introduced by Dickey's approach. Here we use the operator $\triangle$ instead of the operator Γ to define the dKP hierarchy. An obvious advantage [20] of this is that there is a strong connection between the tau functions of dKP and of KP (3.18). With this useful connection, we can use Dickey's convenient approach [17] to obtain the ASvM formula for dKP. In addition, by this connection and the known results of KP's, we can simplify our proof and obtain the difference Fay identity for dKP.

The algebraic structures of additional symmetries and Sato's Bäcklund transformations were also introduced. And ASvM for dKP, which sets up the connection between additional symmetries and Sato's Bäcklund transformations, also constructs the connection between their algebraic structures. In addition, we obtained the ASvM in the form of tau functions, with which we can conveniently get the actions of additional symmetries on the tau function from the actions of additional symmetries on wavefunctions.

Note that when $n=0$, the algebraic structures go back to the algebraic structure of the KP hierarchy. And the corresponding results in the KP hierarchy can be obtained automatically.

With the additional symmetries and ASvM the constraint-dKP can be well defined, which we will try in the near future.

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